Probabilistic Graphical Models Spring 1398 (2019) Midterm Exam	Instructor: B. Nasihatkon	دانشگاه صنعتی خواجه نصیرالدین طوسی K. N. TOOSI UNIVERSITY OF TECHNOLOGY
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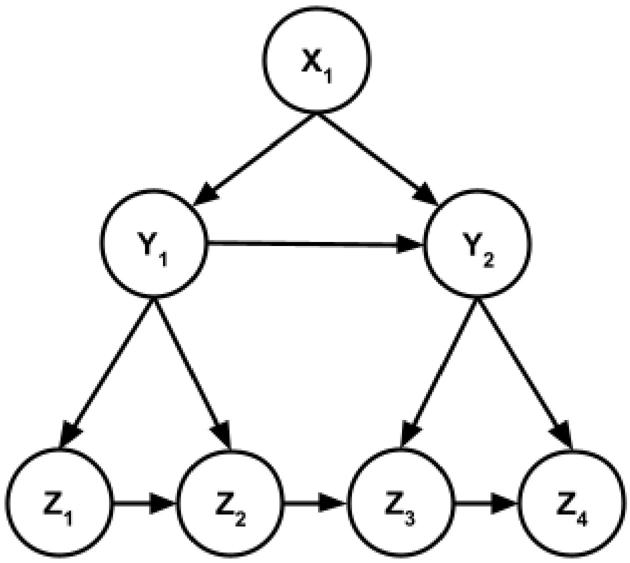
Q1- Bayesian Nets (22 points)

Consider the following Bayesian Network:

A) Write down the joint distribution in terms of the CPDs. (4 points)



$P(X_1, Y_1, Y_2, Z_1, Z_2, Z_3, Z_4)$ =

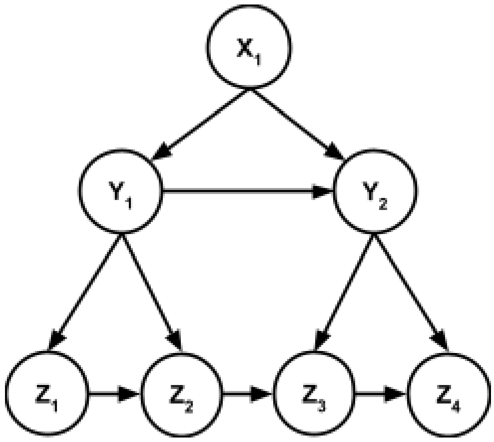
$P(X_1)\, P(Y_1 | X_1)\, P(Y_2 | X_1, Y_1)$
$P(Z_1 | Y_1)\, P(Z_2 | Y_1, Z_1)\, P(Z_3 | Y_2, Z_2)$
$P(Z_4 | Y_2, Z_3)$

B) Which of the following statements are True, and which are False in general (i.e. can be false for some distribution with the above network). In the designated area write the word "True" or an Active Trail rejecting the statement. (18 points).

	True/Active Trail		True/Active Trail
$Y_1 \perp Y_2$	$Y_1 \rightarrow Y_2$	$Z_1 \perp Z_4 \| Z_3, Y_1$	$Z_4 \rightarrow Y_2 \rightarrow Z_3 \rightarrow Z_2 \rightarrow Z_1$
$Y_1 \perp Y_2 \| X_1$	$Y_1 \rightarrow Y_2$	$Z_1 \perp Z_4 \| Z_3, Y_2$	True
$Z_4 \perp Z_2 \| Z_3$	$Z_4 \rightarrow Y_2 \rightarrow Y_1 \rightarrow Z_2$ OR $Z_4 \rightarrow Y_2 \rightarrow Z_3 \rightarrow Z_2$	$X_1 \perp Z_1 \| Y_1$	True
$Z_4 \perp Z_2 \| Z_3, Y_2$	True	$X_1 \perp Z_1 \| Y_1, Z_4$	$X_1 \rightarrow Y_2 \rightarrow Z_4 \rightarrow Z_3$ $\rightarrow Z_2 \rightarrow Z_1$
$Z_4 \perp Z_2 \| Z_3, Y_1$	$Z_4 \rightarrow Y_2 \rightarrow Z_3 \rightarrow Z_2$	$X_1 \perp Z_1 \| Y_1, Z_3$	$X_1 \rightarrow Y_2 \rightarrow Z_3 \rightarrow Z_2 \rightarrow Z_1$
$Z_1 \perp Z_4 \| Z_2$	$Z_4 \rightarrow Y_2 \rightarrow Y_1 \rightarrow Z_1$	$X_1 \perp Z_1 \| Y_1, Z_4, Z_2$	True

Q2 - Probability Queries in Bayes Nets (36 points)

Consider the Bayes nets in Question 1. Write down the following conditional probability queries in terms of the CPDs. Write down full derivations. You may reuse the answers from the previous items. Write it explicitly if you do so.



A) [3pt] $P(Z_4 \mid Y_2, Z_2) = \sum_{Z_3} P(Z_4, Z_3 \mid Y_2, Z_2)$

$= \sum_{Z_3} P(Z_4 \mid Z_3, Y_2, Z_2)\, P(Z_3 \mid Y_2, Z_2)$

$= \sum_{Z_3} P(Z_4 \mid Z_3, Y_2)\, P(Z_3 \mid Y_2, Z_2)$

B) [3pt] $P(Z_1 \mid X_1) = \sum_{Y_1} P(Z_1, Y_1 \mid X_1)$

$= \sum_{Y_1} P(Z_1 \mid Y_1, X_1)\, P(Y_1 \mid X_1) = \sum_{Y_1} P(Z_1 \mid Y_1)\, P(Y_1 \mid X_1)$

C) [4pt] $P(Z_2 \mid Y_1) = \sum_{Z_1} P(Z_2, Z_1 \mid Y_1) = \sum_{Z_1} P(Z_2 \mid Z_1, Y_1)\, P(Z_1 \mid Y_1)$

D) [4pt] $P(Z_2 \mid X_1) = \sum_{Y_1} P(Z_2, Y_1 \mid X_1) = \sum_{Y_1} P(Z_2 \mid Y_1, X_1)\, P(Y_1 \mid X_1) = \sum_{Y_1} P(Z_2 \mid Y_1)\, P(Y_1 \mid X_1)$

The last equality is due to the fact that $Z_2 \perp X_1 \mid Y_1$ and $P(Z_2 \mid Y_1)$ is derived in part C.

E) [4pt] $P(Y_2 \mid X_1) = \sum_{Y_1} P(Y_2, Y_1 \mid X_1) = \sum_{Y_1} P(Y_2 \mid Y_1, X_1)\, P(Y_1 \mid X_1)$

F) [5pt] $P(Z_2, Y_2 \mid X_1) = \sum_{Y_1} P(Z_2, Y_2, Y_1 \mid X_1) = \sum_{Y_1} P(Z_2, Y_2 \mid Y_1, X_1)\, P(Y_1 \mid X_1)$

Now, notice that $Z_2 \perp Y_2 \mid Y_1, X_1$, thus the above is

$= \sum_{Y_1} P(Z_2 \mid Y_1, X_1)\, P(Y_2 \mid Y_1, X_1)\, P(Y_1 \mid X_1) = \sum_{Y_1} P(Z_2 \mid Y_1)\, P(Y_2 \mid Y_1, X_1)\, P(Y_1 \mid X_1)$,

where $P(Z_2 \mid Y_1)$ is calculated in part C.

G) [6pt] $P(Z_3 \mid X_1) = \sum_{Z_2} \sum_{Y_2} P(Z_3, Z_2, Y_2 \mid X_1) = \sum_{Z_2} \sum_{Y_2} P(Z_3 \mid Z_2, Y_2, X_1)\, P(Z_2, Y_2 \mid X_1)$

$= \sum_{Z_2} \sum_{Y_2} P(Z_3 \mid Z_2, Y_2)\, P(Z_2, Y_2 \mid X_1)$

where $P(Z_2, Y_2 \mid X_1)$ is given in part F.

H) [7pt] $P(Z_4 \mid X_1) = \sum_{Z_3}\sum_{Z_2}\sum_{Y_2} P(Z_4, Z_3, Z_2, Y_2 \mid X_1) = \sum_{Z_3}\sum_{Z_2}\sum_{Y_2} P(Z_4 \mid Z_3, Z_2, Y_2, X_1)\, P(Z_3, Z_2, Y_2 \mid X_1)$

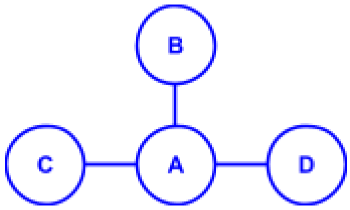
$\sum_{Z_3}\sum_{Z_2}\sum_{Y_2} P(Z_4 \mid Z_3, Y_2) P(Z_3 \mid Z_2, Y_2, X_1)\, P(Z_2, Y_2 \mid X_1) = \sum_{Z_3}\sum_{Z_2}\sum_{Y_2} P(Z_4 \mid Z_3, Y_2) P(Z_3 \mid Z_2, Y_2)\, P(Z_2, Y_2 \mid X_1)$

where $P(Z_2, Y_2 \mid X_1)$ is given in part F.

Q3- Markov Nets (42 points)

Consider the Gibbs distribution below defined on binary random variables $A, B, C, D \in \{0, 1\}$.

$$P(A, B, C, D) = \frac{1}{Z} \exp(AB - AC + C + A(1 - 2D))$$



$$= \frac{1}{Z} \exp(AB) \exp(-AC) \exp(C) \exp(A(1 - 2D))$$

A) Draw the corresponding Markov network. (3 points)

B) Perform variable elimination in the order B, C, D, A. Write down the full derivations for each elimination step. What is the value of the partition function Z? Write Z in terms of the Euler's (Napier's) number $e = \exp(1)$. (12 points)

$\tilde{P}(A, B, C, D) = \exp(AB) \exp(-AC) \exp(C) \exp(A(1 - 2D))$

Eliminate B: $\tau_b(A) = \sum_{B=0}^{1} \exp(AB) = \exp(0) + \exp(A) = 1 + \exp(A)$

$\tilde{P}(A, C, D) = \tau_b(A) \exp(-AC) \exp(C) \exp(A(1 - 2D))$

Eliminate C: $\tau_c(A) = \sum_{C=0}^{1} \exp(-AC) \exp(C)$

$= \exp(0) \exp(0) + \exp(-A) \exp(1) = 1 + \exp(1 - A)$

$\tilde{P}(A, D) = \tau_b(A) \tau_c(A) \exp(A(1 - 2D))$

Eliminate D: $\tau_d(A) = \sum_{D=0}^{1} \exp(A\,(1 - 2D)) = \exp(A) + \exp(-A)$

$\tilde{P}(A) = \tau_b(A)\,\tau_c(A)\,\tau_d(A)$

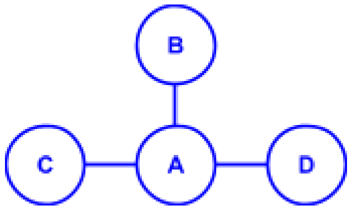
Eliminate A:

$\tau_a = \sum_{A=0}^{1} \tau_b(A)\,\tau_c(A)\,\tau_d(A) = \sum_{A=0}^{1} (1 + \exp(A))\,(1 + \exp(1 - A))\,(\exp(A) + \exp(-A))$

$= (2)\,(1 + e)\,(2) + (1 + e)\,(2)\,(e + e^{-1})$

$= 2\,(1 + e)\,(2 + e + e^{-1}) = \sum_{A=0}^{1}\sum_{B=0}^{1}\sum_{C=0}^{1}\sum_{D=0}^{1} \tilde{P}(A, B, C, D) = Z$

C) Draw the corresponding induced graph of part B. (3 points)



D) Derive P(D|A) using the results of the above elimination steps. (4 points)

$P(D \mid A) = P(D, A) / P(A) = \tilde{P}(D, A) / \tilde{P}(A)$

$= \tau_b(A)\,\tau_c(A)\,\exp(A\,(1 - 2D)) / (\tau_b(A)\,\tau_c(A)\,\tau_d(A)) = \exp(A\,(1 - 2D)) / \tau_d(A)$

$= \exp(A\,(1 - 2D)) / (\exp(A) + \exp(-A)) = \exp(1 - 2D) / (1 + \exp(-2A))$

E) Derive P(C | D, A) using the results of the above elimination steps. (4 points)

$P(C \mid D, A) = P(C, D, A) / P(D, A) = \tilde{P}(C, D, A) / \tilde{P}(D, A)$

$= \tau_b(A)\,\exp(-AC)\,\exp(C)\,\exp(A\,(1 - 2D)) / (\tau_b(A)\,\tau_c(A)\,\exp(A\,(1 - 2D)))$

$= \exp(-AC)\,\exp(C) / \tau_c(A) = \exp(C - AC) / (1 + \exp(1 - A))$

Notice that $P(C \mid D, A)$ is not a function of D. This is expected because $C \perp D \mid A$, and thus $P(C \mid D, A) = P(C \mid A)$

F) Repeat the variable elimination, this time in the order A, B, C, D. Confirm that the value of the partition function Z is the same as part B. (12 points)

$\tilde{P}(A, B, C, D) = \exp(AB)\,\exp(-AC)\,\exp(C)\,\exp(A\,(1 - 2D))$

Eliminate A: $\tau_a(B, C, D) = \sum_{A=0}^{1} \exp(AB)\,\exp(-AC)\,\exp(C)\,\exp(A\,(1 - 2D))$

$= \exp(0)\,\exp(0)\,\exp(C)\,\exp(0) + \exp(B)\,\exp(-C)\,\exp(C)\,\exp(1 - 2D) = \exp(C) + \exp(B)\,\exp(1$

$\tilde{P}(B, C, D) = \tau_a(B, C, D) = \exp(C) + \exp(B)\,\exp(1 - 2D)$

Eliminate B:

$\tau_b(C, D) = \sum_{B=0}^{1} (\exp(C) + \exp(B)\,\exp(1 - 2D)) = 2\exp(C) + (1 + e)\,\exp(1 - 2D)$

$$\tilde{P}(C, D) = \tau_b(C, D) = 2\exp(C) + (1 + e)\exp(1 - 2D)$$

Eliminate C:

$$\tau_c(D) = \sum_{C=0}^{1} (2\exp(C) + (1 + e)\exp(1 - 2D)) = 2(1 + e) + 2(1 + e)\exp(1 - 2D)$$
$$= 2(1 + e)(1 + \exp(1 - 2D))$$

$$\tilde{P}(D) = \tau_c(D) = 2(1 + e)(1 + \exp(1 - 2D))$$

Eliminate D: $\tau_d = \sum_{D=0}^{1} 2(1 + e)(1 + \exp(1 - 2D)) = 2(1 + e)(2 + e + e^{-1}) = Z$

G) Draw the corresponding induced graph of part F. (4 points)